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Increasing and Decreasing Operators on Complete Lattices

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The (isotone) map $f: X \to X$ is an increasing (decreasing) operator on the poset X if $f(x) \leqslant f^2(x)$ ($f^2(x) \leqslant f(x)$, resp.) holds for each $x \in X$. Properties of increasing (decreasing) operators on complete lattices are studied and shown to extend and clarify those of closure (resp. anticlosure) operators. The notion of the decreasing closure, $\bar{f}$, (the increasing anticlosure, $\underline{f}$,) of the map $f: X \to X$ is introduced extending that of the transitive closure, $\hat{f}$, of f. f, $\bar{f}$, and $\underline{f}$ are all shown to have the same set of fixed points. Our results enable us to solve some problems raised by H. Crapo. In particular, the order structure of $\mathscr{H}(X)$, the set of retraction operators on X is analyzed. For X a complete lattice $\mathscr{H}(X)$ is shown to be a complete lattice in the pointwise partial order. We conclude by claiming that it is the increasing–decreasing character of the identity maps which yields the peculiar properties of Galois connections. This is done by defining a u–v connection between the posets X and Y, where $u: X \to X$ ($v: Y \to Y$) is an increasing (resp. decreasing) operator to be a pair f, g of maps $f: X \to Y$, $g: Y \to X$ such that $gf \geqslant u$, $fg \leqslant v$. It is shown that the whole theory of Galois connections can be carried over to u–v connections.

1

Let X, Y denote partially ordered sets (posets). Y^X is the set of all isotone (order preserving) maps $f: X \to Y$ partially ordered by the pointwise partial order, i.e., $f_1 \leqslant f_2 \in Y^X$ when $f_1(x) \leqslant f_2(x)$ holds for each $x \in X$. All maps considered in this paper are isotone and unless otherwise specified they belong to X^X. The map $f: X \to X$ is an *increasing* (*decreasing*) operator on X when it satisfies $f \leqslant f^2$ ($f^2 \leqslant f$ resp.). $\mathscr{U}(X)$($\mathscr{V}(X)$) denotes the set of all increasing (decreasing resp.) operators on X. The map $h: X \to X$ is a *retraction* operator on X [6] if $h \in \mathscr{U}(X) \cap \mathscr{V}(X) = \mathscr{H}(X)$. Obviously, h is a retraction iff $h = h^2$, i.e., iff h is an idempotent in the po-semigroup (with respect to composition) of maps on X. A retraction operator h satisfying $h \geqslant i$ ($h \leqslant i$), where i is the identity operator is a *closure* (*anticlosure* resp.) operator. The sets $\mathscr{U}(X)$, $\mathscr{V}(X)$ and $\mathscr{H}(X)$ are all partially ordered, as subsets of X^X, by the pointwise partial order.

For each map $f\colon X \to X$, let $X_f \subseteq X$ denote the set of fixed points of f, $X_f = \{\bar{x} \mid \bar{x} = f(\bar{x})\}$. Tarski [11] has shown that when X is a complete lattice $X_f \subseteq X$ is not empty and is a complete lattice in the partial order induced by X. In particular, when h is a retraction on the complete lattice X then $X_h = h(X)$, the image set of h, is a complete lattice (see also [1]). As shown in [6] the converse holds as well and thus $X' \subseteq X$ is a *retract* of X, i.e., $X' = h(X)$ for some retraction h, iff X' is a complete lattice in the partial order induced by X. As remarked in [6], the retraction on X whose image is X' is by no means unique.

The pair $\langle f, g \rangle$ of (isotone) maps $f\colon X \to Y$, $g\colon Y \to X$ is a (dual) *Galois connection* between X and Y [8], when $gf \geqslant i_X$, $fg \leqslant i_Y$ are satisfied, where i_X, i_Y are the identity maps on X, Y, resp. The properties of Galois connections are well known [4, 5, 9]. In particular one has $f = fgf$, $g = gfg$, where gf is a closure operator on X and fg an anticlosure operator on Y. The maps f, g in a Galois connection determine each other uniquely. When X, Y are complete lattices then $f(X)$ and $g(Y)$ are isomorphic complete $\bigvee$-, $\bigwedge$-sublattices of Y, X, resp. Galois connections can also be composed. When X, Y are complete lattices the set $X \circledast Y \subseteq Y^X$ of all *residuated* maps [5], i.e., all those maps $f\colon X \to Y$ which are part of a Galois connection $\langle f, g \rangle$ between X and Y, is a complete $\bigvee$-sublattice of Y^X (see [10], where $X \otimes Y^D$ stands for $X \circledast Y$).

A generalization of the notion of a Galois connection is suggested in [6], replacing the closure and anticlosure operators in the original definition by retraction operators. Thus the pair $\langle f, g \rangle$ of maps $f\colon X \to Y$, $g\colon Y \to X$ is a *connection* when $f = fgf$ and $g = gfg$. It follows that $gf\colon X \to X$ $fg\colon Y \to Y$ are both retractions. Also when X, Y are complete lattices $f(X) \subseteq Y$ and $g(Y) \subseteq X$ are isomorphic complete lattices in the induced order. However, in such a connection the maps f, g do not uniquely determine each other. Also, the composite of two connections is not in general a connection. Another approach is suggested in [2]. Both approaches take one of the outcomes of the original definition of a Galois connection as their starting point.

2

In this paper we study the notions of increasing and decreasing operators and show that their properties closely resemble those of closure and anticlosure operators, resp. The notion of the *decreasing closure* $\bar{f}$ (*increasing anticlosure* $\underline{f}$) of the map $f\colon X \to X$ is introduced and shown to extend that of the *transitive closure* $\hat{f}$ [3] (*transitive anticlosure* $\check{f}$, resp.) of f. A close relationship is found between increasing (and analogously decreasing) operators and retractions associated with them. Among other things, this sheds light on the order structure of $\mathscr{H}(X)$, enabling us to solve some problems raised in [6]. Following the above mentioned ideas, we call a pair $\langle f, g\rangle$ of maps $f\colon X \to Y$, $g\colon Y \to X$ a *u–v connection* between X and Y, where $u\colon X \to X$ is an increasing operator and $v\colon Y \to Y$ is a decreasing operator, when $gf \geqslant u$ and $fg \leqslant v$. We then show that the whole theory of (dual) Galois ($i_X - i_Y$) connections can be carried over to u–v connections. Thus it is the increasing–decreasing character of the identity maps i_X, i_Y rather than their idempotency which is crucial in the definition of a Galois connection.

From now on X will denote a complete lattice when not otherwise specified. With each map $f\colon X \to X$ we associate its decreasing closure $V(f) = \bar{f} \in \mathscr{V}(X)$ and its increasing anticlosure $U(f) = \underline{f} \in \mathscr{U}(X)$, where $\underline{f} \leqslant f \leqslant \bar{f}$. $\bar{f}$ is the least decreasing map greater than f, and $\underline{f}$ is the greatest increasing map smaller than f. Of utmost importance is the fact that $X_f = X_{\bar{f}} = X_{\underline{f}}$. When g is an increasing operator then $\bar{g}$ is a retraction, also denoted by $\bar{h}_g$, with X_g as its image set. Similarly, when f is a decreasing operator then $\underline{f}$ is a retraction, also denoted by $\underline{h}_f$, with X_f as its image set. The maps $\bar{h}_g$, $\underline{h}_f$ are called the *retractions associated with the* (*increasing*, *decreasing*, resp.) *operators* g, f. The maps VU, $UV\colon X^X \to X^X$ defined by $VU(f) = V(\underline{f}) = \bar{h}_{\underline{f}}$, $UV(f) = U(\bar{f}) = \underline{h}_{\bar{f}}$ for each $f \in X^X$, are retraction operators on the complete lattice X^X both having $\mathscr{H}(X)$ as their image set. This proves that when X is a complete lattice, $\mathscr{H}(X)$ is a complete lattice. For X a finite lattice this result was proved in [6]. We bring an example to show that when X is a (noncomplete) lattice $\mathscr{H}(X)$ is not in general a lattice, thus answering [6, Problem 5]. When $\{h_i\} \subseteq \mathscr{H}(X)$ is a given set of retractions on X then $\bigvee_{\mathscr{H}(X)} h_i$ is the retraction associated with the increasing map $\bigvee h_i$ and has $X_{\vee h_i}$ as its image set. Similarly, since $\bigwedge h_i$ is decreasing, $\bigwedge_{\mathscr{H}(X)} h_i$ is the retraction associated with $\bigwedge h_i$ and its image set is $X_{\wedge h_i}$.

The analogy between closure (anticlosure) operators on the one hand and increasing (decreasing, resp.) operators on the other hand is studied. In particular, some well-known properties of closure operators and their composites [7, 12], are shown to be possessed by increasing operators and their composites. For example, it is shown that if the maps $f_1, f_2 : X \to X$ satisfy $f_1 \vee f_2 \leqslant f_1 f_2$ and $f_1 \vee f_2 \leqslant f_2 f_1$, then both $f_1 f_2$, $f_2 f_1$ and $f_1 \vee f_2$ are increasing operators and the retractions associated with them satisfy $\overline{f_1 f_2} = \overline{f_2 f_1} = \overline{f_1 \vee f_2}$ and $X_{f_1 \vee f_2} = X_{f_1 f_2} = X_{f_2 f_1} = X_{f_1} \cap X_{f_2}$. Thus $f_1 f_2$ is a retraction operator iff $f_1 f_2 = \overline{f_1 \vee f_2}$; if $f_1 \vee f_2$ is a retraction operator then $f_1 f_2 = f_2 f_1 = f_1 \vee f_2$. If h_1, h_2 are retractions satisfying $h_1 \vee h_2 \leqslant h_1 h_2$, $h_1 \vee h_2 \leqslant h_2 h_1$ then both $h_1 h_2$ and $h_2 h_1$ are retractions iff $h_1 h_2 = h_2 h_1$. This gives a partial answer to [6, Problem 6].

The notion of a u–v connection defined above is found to be a natural extension of the notion of a Galois $(i_X - i_Y)$ connection. Thus it follows that if $\langle f, g\rangle$ is a u–v connection between the posets X, Y, then $\tilde{f} = vfu = vfgfu$, $\tilde{g} = ugv = ugfgv$; the maps $\tilde{g}\tilde{f} = ugvfu \geqslant u$, $\tilde{f}\tilde{g} = vfugv \leqslant v$ are retractions on X, Y, resp., with $\tilde{f} = \tilde{f}\tilde{g}\tilde{f}$, and $\tilde{g} = \tilde{g}\tilde{f}\tilde{g}$. Thus $\langle \tilde{f}, \tilde{g}\rangle$ is a u–v connection, which is also a connection in Crapo's sense. Relative to the given maps u, v, the maps $\tilde{f}$, $\tilde{g}$, "generated" by f, g, determine each other uniquely. u–v connections can be composed provided some natural restrictions are imposed on the maps involved. Also, when X, Y are complete lattices, the poset $X \circledast_{(u,v)} Y \subseteq Y^X$ of all maps $f\colon X \to Y$ which participate in a u–v connection between X and Y is a complete $\bigvee$-sublattice of Y^X.

3

In this section we introduce the notions of the decreasing closure and the increasing anticlosure.

THEOREM 1.3. *Let X be a complete lattice. Then $\mathscr{U}(X)$, $(\mathscr{V}(X))$, the set of all increasing (decreasing, resp.) operators on X is a complete $\bigvee$- ($\bigwedge$-resp.) sublattice of X^X in the pointwise partial order.*

Proof. The map $I\colon X \to X$ satisfying $I(x) = 1$ for each $x \in X$ is the greatest element of $\mathscr{V}(X)$. Also, for $\{f_i\} \subseteq \mathscr{V}(X)$ we have $(\bigwedge f_i)^2(x) = \bigwedge f_i(\bigwedge f_i(x)) \leqslant \bigwedge f_i^2(x) \leqslant \bigwedge f_i(x) = (\bigwedge f_i)(x)$, proving that $\bigwedge f_i \in \mathscr{V}(X)$. Thus [4, p. 112] $\mathscr{V}(X)$ is a complete $\bigwedge$-sublattice of X^X. The proof that $\mathscr{U}(X)$ is a complete $\bigvee$-sublattice of X^X is analogous.

With each map $f\colon X \to X$, where X is a complete lattice we now associate two maps denoted by $V(f) = \bar{f}$, $U(f) = \underline{f}$ as follows.

$$\begin{aligned} V(f) &= \bar{f} = \bigwedge \{f_i \mid f_i \in \mathscr{V}(X), f \leqslant f_i\}, \\ U(f) &= \underline{f} = \bigvee \{f_{i'} \mid f_{i'} \in \mathscr{U}(X), f_{i'} \leqslant f\}. \end{aligned} \tag{1}$$

COROLLARY 1.3. *For each map $f\colon X \to X$ where X is a complete lattice the map $\bar{f} = V(f)$ is a decreasing operator, the map $\underline{f} = U(f)$ is an increasing operator and $\underline{f} \leqslant f \leqslant \bar{f}$ holds. $\bar{f}$ is the least decreasing operator greater than f, while $\underline{f}$ is the greatest increasing operator smaller than f.*

We call $\bar{f} = V(f)$ the *decreasing closure* of f and $\underline{f} = U(f)$ the *increasing anticlosure* of f. Since (1) implies that if $f_1 \leqslant f_2 \in X^X$ then $\bar{f}_1 \leqslant \bar{f}_2$ and $\underline{f}_1 \leqslant \underline{f}_2$, and also that for $f \in \mathscr{V}(X)$, $\bar{f} = f$, while for $f \in \mathscr{U}(X)$ $\underline{f} = f$ we have:

THEOREM 2.3. *For a complete lattice X the maps $V\colon X^X \to X^X$, $U\colon X^X \to X^X$ defined by $V(f) = \bar{f}$, $U(f) = \underline{f}$ when $f \in X^X$, are closure, anticlosure operators, resp., on X^X with $\mathscr{V}(X)$, $\mathscr{U}(X)$, resp., as their image sets.*

The following theorem plays a central role in what follows.

THEOREM 3.3. *Let X be a complete lattice. If $f\colon X \to X$ is a decreasing operator then $\underline{f} = U(f)$, its increasing anticlosure, is a retraction operator denoted also by $\underline{h}_f$. Similarly, when $g\colon X \to X$ is an increasing operator then $\bar{g} = V(g)$, its decreasing closure, is a retraction operator denoted also by $\bar{h}_g$.*

Proof. Since f is decreasing and $\underline{f}$ is increasing with $\underline{f} \leqslant f$ we get $\underline{f} \leqslant \underline{f}^2 \leqslant f^2 \leqslant f$. Since $\underline{f}^2 \leqslant (\underline{f}^2)^2$, $\underline{f}^2$ is itself increasing and smaller than f. By (1), $\underline{f}^2 \leqslant \underline{f}$ follows, yielding $\underline{f} = \underline{f}^2 = \underline{h}_f \leqslant f$. The proof that when g is increasing then $\bar{g} = \bar{g}^2 = \underline{h}_g \geqslant g$ is analogous.

COROLLARY 2.3. *For each decreasing operator f on the complete lattice X there exists a retraction, namely $\underline{h}_f$, such that $\underline{h}_f \leqslant f$ and h_f is the greatest among all retractions smaller than f. Similarly, for each increasing operator g on X there exists a retraction, namely $\bar{h}_g$, such that $g \leqslant \bar{h}_g$ and $\bar{h}_g$ is the smallest among all retractions greater than g. $\underline{h}_f(\bar{h}_g)$ is the retraction associated with the decreasing (increasing, resp.) map $f(g$, resp.).*

We now have the following result, proved in [6] for the finite case (see [6, Problem 5]).

THEOREM 4.3. *Let X be a complete lattice. Then $\mathscr{H}(X)$, the set of retractions on X, is a complete lattice in the pointwise partial order.*

Proof. By Theorems 2.3, 3.3 it follows that the map $UV\colon X^X \to X^X$ defined by $UV(f) = U(\bar{f}) = \underline{h}_{\bar{f}}$ for $f \in X^X$ is isotone with $UV(f) \in \mathscr{H}(X)$. Since for $f \in \mathscr{H}(X)$ $f = UV(f)^-$, it follows that UV is a retraction operator on the complete lattice X^X. $\mathscr{H}(X)$ is thus a retract of X^X and therefore a complete lattice in the pointwise order. Notice that the retraction $VU\colon X^X \to X^X$ defined by $VU(f) = V(\underline{f}) = \bar{h}_{\underline{f}}$ for $f \in X^X$ could be used to the same effect.

In answer to a problem posed by Crapo we now show that when X is a (noncomplete) lattice, $\mathscr{H}(X)$ need not in general be a lattice. We need the following.

LEMMA 1.3. *Let $f\colon X \to X$ be a decreasing operator on the poset X. If there exists a retraction $h\colon X \to X$ such that $h \leqslant f$ then f has a fixed point.*

Proof. By $h = h^2 \leqslant f$ we get $f^2h(x) \leqslant fh(x) = fh^2(x) \leqslant f^2h(x)$ for each $x \in X$. Thus letting $\bar{x} = fh(x)$, $f(\bar{x}) = \bar{x}$ follows proving the lemma.

· EXAMPLE 1.3. Let X be the noncomplete chain of integers $\{..., -2, -1, 0, 1, 2,...\}$. The maps $h_1 : X \to X$, $h_2 : X \to X$ given by

$$h_1(x) = 2n - 1, \quad x = 2n, \qquad h_2(x) = 2n, \quad x = 2n,$$
$$ = 2n - 1, \quad x = 2n - 1, \qquad = 2n - 2, \quad x = 2n - 1,$$

are both retractions on X. We show that there does not exist a retraction h such that $h \leqslant h_1$, $h \leqslant h_2$. Actually, if such a retraction exists then $h \leqslant h_1 \wedge h_2$ must hold. Since $(h_1 \wedge h_2)(n) = n - 1$, $h_1 \wedge h_2$ is a decreasing map and has no fixed points. By Lemma 1.3 we have arrived at a contradiction and thus $\mathscr{H}(X)$ is not a lattice.

4

In order to get a better understanding of the order structure of $\mathscr{H}(X)$ and also of its relations with $\mathscr{U}(X)$ and $\mathscr{V}(X)$ (Theorem 3.3) we are now going to study the complete lattice X_f of fixed points of the map $f: X \to X$. First we show that f, its decreasing closure $\bar{f}$, and its increasing anticlosure $\underline{f}$ all possess the same set of fixed points.

THEOREM 1.4. *Let X be a complete lattice and let $f: X \to X$ be a given map. Then $X_f = X_{\bar{f}} = X_{\underline{f}}$.*

Proof. If $f(x_0) = x_0$ then for $x \leqslant x_0$, $f(x) \leqslant f(x_0) = x_0$ holds. The map $h_0 : X \to X$ satisfying $h_0(x) = x_0$ for $x \leqslant x_0$ and $h_0(x) = 1$ otherwise is therefore a retraction on X and $f \leqslant h_0$. By (1) it follows that $f \leqslant \bar{f} \leqslant h_0$, hence $x_0 = f(x_0) \leqslant \bar{f}(x_0) \leqslant h_0(x_0) = x_0$, implying that $\bar{f}(x_0) = x_0$. Consequently $X_f \subseteq X_{\bar{f}}$. Conversely, if $\bar{f}(x_1) = x_1$ then $f(x_1) \leqslant \bar{f}(x_1) = x_1$ and so when $x \leqslant x_1$ then $f(x) \leqslant x_1$. The map $h_1 : X \to X$ which satisfies $h_1(x) = f(x_1)$ for $x \leqslant x_1$ and $h_1(x) = 1$ otherwise is a retraction operator and $f \leqslant h_1$. By (1), $f \leqslant \bar{f} \leqslant h_1$, and $f(x_1) \leqslant \bar{f}(x_1) = x_1 \leqslant h_1(x_1) = f(x_1)$, implying that $x_1 = f(x_1)$. Hence $X_{\bar{f}} \subseteq X_f$ and equality follows. $X_f = X_{\underline{f}}$ is proved analogously.

By Theorems 3.3, 1.4 we now have:

COROLLARY 1.4. *Let $f: X \to X$ be a map on the complete lattice X. Then both retraction operators $UV(f) = \underline{h}_{\bar{f}}$, $VU(f) = \bar{h}_{\underline{f}} : X \to X$ associated with f have X_f, the complete lattice of fixed points of f, as their image set.*

In particular, when f is a decreasing (increasing) operator then $\underline{h}_f$ ($\bar{h}_f$, resp.), the retraction associated with f, satisfies $\underline{h}_f \leqslant \cdots \leqslant f^n \leqslant \cdots \leqslant f^2 \leqslant f(f \leqslant f^2 \leqslant \cdots \leqslant f^n \leqslant \cdots \leqslant \bar{h}_f$ – resp.) and $\underline{h}_f(X)$ ($\bar{h}_f(X)$, resp.) $= X_f$. The relationship between decreasing (increasing) operators and the retraction operators associated with them is clarified by Theorem 2.4, and is based on the following result due to Tarski [11].

LEMMA 1.4 [11]. *Let $f(x_0) \leqslant x \leqslant (x_o \leqslant f(x_o))$ for some $x_0 \in X$ where f is a map on the complete lattice X. Then there exists a fixed point $\bar{x} = f(\bar{x})$ such that $\bar{x} \leqslant x_0$ ($x_0 \leqslant \bar{x}$, resp.) and $\bar{x}$ is the greatest (least, resp.) fixed point of f which is smaller (greater, resp.) than x_0.*

Proof. Since $0 \leqslant f(0) \leqslant f(x_0) \leqslant x_0$ it follows that the set $S_{x_0} = \{x_i \mid x_i \leqslant f(x_i), x_i \leqslant x_0\}$ is not empty. Now let $\bar{x} = \bigvee \{x_i \mid x_i \in S_{x_0}\}$. Then $\bar{x} = \bigvee x_i \leqslant \bigvee f(x_i) \leqslant f(\bigvee x_i) = f(\bar{x})$, and $\bar{x} \leqslant x_0$. Thus $\bar{x} \in S_{x_0}$. Since $f(\bar{x}) \leqslant f(f(\bar{x}))$ and also $\bar{x} \leqslant f(\bar{x}) \leqslant f(x_0) \leqslant x_0$ it follows that $f(\bar{x}) \in S_{x_0}$ and thus $f(\bar{x}) \leqslant \bar{x}$, implying that $\bar{x} = f(\bar{x})$. The fact that $\bar{x}$ is the greatest among all fixed points less than x_0 is obvious. The other part of the lemma is proved analogously.

We can now prove:

THEOREM 2.4. *Let X be a complete lattice and let $f\colon X \to X$($g\colon X \to X$) be a decreasing (increasing, resp.) operator. Then for each $x \in X$:*

$$\underline{h}_f(x) = \bigvee_{X_f} \{x_i \mid x_i = f(x_i) \leqslant f(x)\} \leqslant f(x),$$
$$\left(\bar{h}_g(x) = \bigwedge_{X_g} \{x_i \mid x_i = g(x_i) \geqslant g(x)\} \geqslant g(x), \text{resp.}\right). \tag{2}$$

Proof. Since f is a decreasing operator $f(f(x) \leqslant f(x))$ holds for each $x \in X$. By Lemma 1.4 there exists $\bar{x} \in X$ such that $\bar{x} = f(\bar{x}) \leqslant f(x)$ and $\bar{x}$ is the greatest fixed point of f smaller than $f(x)$. Let $h^*\colon X \to X$ be defined by $h^*(x) = \bar{x}$. It follows easily that h^* is isotone and that for $x \in X_f$ $h^*(x) = x$. Thus h^* is a retraction satisfying $h^* \leqslant f$. By (1) and Corollary 2.3, $h^* \leqslant \underline{h}_f$ follows. To prove the converse inclusion notice that $\underline{h}_f = f \leqslant f$ (Theorem 3.3) and $\underline{h}_f(X) = X_f$ (Theorem 1.4). Hence $\underline{h}_f(x)$ is a fixed point of f which is less than $f(x)$, and thus $\underline{h}_f(x) \leqslant \bar{x} = h^*(x)$ follow implying $\underline{h}_f \leqslant h^*$. Consequently $\underline{h}_f(x) = h^*(x) = \bigvee_{X_f}\{x_i \mid x_i = f(x_i) \leqslant f(x)\}$, proving the first part of the theorem. The proof of the second part is analogous.

By Theorems 1.3, 1.4 and Corollary 2.3 a better understanding of te order structure of the complete lattice (Theorem 4.3) $\mathscr{H}(X) = \mathscr{U}(X) \cap \mathscr{V}(X)$ can be gained:

COROLLARY 2.4. *Let $\{h_i\}$ be a given set of retraction operators on the complete lattice X. Then $\bigwedge_{\mathscr{H}(X)} h_i$ is the retraction $\underline{\bigwedge h_i}$ associated with the decreasing operator $\bigwedge h_i$, and has the set of fixed points of $\bigwedge h_i$ as its image set. Similarly, $\bigvee_{\mathscr{H}(X)} h_i$ is the retraction $\overline{\bigvee h_i}$ associated with the increasing map $\bigvee h_i$ and has the set of fixed points of $\bigvee h_i$ as its image set.*

Another corollary of Theorem 1.4 is:

THEOREM 3.4. *Let $f, f^* \in X^X$ be given maps on the complete lattice X such that $f \leqslant f^* \leqslant \bar{f}$ ($\underline{f} \leqslant f^* \leqslant f$). Then $\bar{f} = \overline{f^*}$ ($\underline{f} = \underline{f^*}$, resp.) and $X_f = X_{f^*}$.*

Proof. By Theorem 2.3 we get that $\bar{f} \leqslant \overline{f^*} \leqslant \bar{\bar{f}} = \bar{f}$ and $\bar{f} = \overline{f^*}$ follows. By Theorem 1.4 $X_f = X_{\bar{f}} = X_{\overline{f^*}} = X_{f^*}$. The other part of the theorem is similarly proved.

Noting that if $f\colon X \to X$ is an increasing (decreasing) operator then $f \leqslant f^2 \leqslant \cdots \leqslant f^n \leqslant \cdots \leqslant \bar{h}_f = \bar{f}$ ($\underline{h}_f = \underline{f} \leqslant \cdots \leqslant f^n \leqslant \cdots \leqslant f^2 \leqslant f$ resp.), Theorem 3.4 yields:

COROLLARY 3.4. *If $f\colon X \to X$ is an increasing (decreasing) operator on the complete lattice X then $\bar{h}_f = \bar{h}_{f^n}$ ($\underline{h}_f = \underline{h}_{f^n}$, resp.) and $X_f = X_{f^n}$ for each number n.*

Thus decreasing operators and increasing operators $f\colon X \to X$ share the property (self-evident for retractions) that f and all its powers have the same set of fixed points.

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When the inclusion $g \leqslant f$ holds where $g\colon X \to X$ is increasing and $f\colon X \to X$ is decreasing, then $g \leqslant g^2 \leqslant \cdots \leqslant g^n \leqslant \cdots \cdots \leqslant f^n \leqslant \cdots \leqslant f^2 \leqslant f$. As follows from the next theorem, such a pair of maps is always separated by its associated retractions, namely, $g \leqslant \bar{h}_g \leqslant \underline{h}_f \leqslant f$ holds. Also the subsemigroup (with respect to composition) S generated by $\{f, g\}$ in that case consists of $\{f^n\} \cup \{g^n\} \cup \{fg, gf, fgf, gfg\}$, the last four maps being the only idempotents of S.

THEOREM 1.5. *Let X be a complete lattice and let f, g be decreasing, increasing operators, resp., on X such that $g \leqslant f$. Then* (i) $g \leqslant \bar{h}_g \leqslant \underline{h}_f \leqslant f$; (ii) *the maps fg, gf, fgf, gfg are all retraction operators with $\bar{h}_g \leqslant gfg = gf \bigwedge_{\mathscr{H}(X)} fg$, $gf \bigvee_{\mathscr{H}(X)} fg = fgf \leqslant \underline{h}_f$;* (iii) *$f^n g^m = fg$, $g^n f^m = gf$ for $n, m \geqslant 1$.*

Proof. Using the istonicity of f, g together with $g \leqslant f$ we get $fg \leqslant fg^{n+m-1} \leqslant f^n g^m \leqslant f^{n+m-1}g \leqslant fg$ proving that $f^n g^m = fg$. $g^n f^m = gf$ follows similarly and (iii) is proved. Since $gf = g^3 f \leqslant gfgf \leqslant gf^3 = gf$ and since $g \leqslant g^2 \leqslant gf \leqslant f^2 \leqslant f$ it follows that gf is a retraction and $g \leqslant gf \leqslant f$. By Corollary 2.3, $\bar{h}_g \leqslant gf \leqslant \underline{h}_f$ follows, proving (i). The idempotency of fg is shown similarly. Using (iii) we get $(fgf)^2 = fgf^2gf = fgfgf = fgf$. $(gfg)^2 = gfg$ is similarly proved. Now gfg is a retraction satisfying $gfg \leqslant gf$, $gfg \leqslant fg$. If some retraction $h: X \to X$ satisfies $h \leqslant gf, h \leqslant fg$ then $h = h^2 \leqslant gffg = gfg$ (by (iii)); hence $gfg = gf \bigwedge_{\mathscr{H}(X)} fg$. Since $\bar{h}_g \leqslant gf, \bar{h}_g \leqslant fg$ it follows that $\bar{h}_g \leqslant gfg$, proving the first statement of (ii). The other statement of (ii) is proved similarly.

COROLLARY 1.5. *Let $f: X \to X$ be given where X is a complete lattice then $VU(f) = \bar{h}_{\underline{f}} \leqslant \underline{h}_{\bar{f}} = UV(f)$.*

Since $U(f) = \underline{f} \leqslant f \leqslant \bar{f} = V(f)$ and $\underline{f}, \bar{f}$ are increasing, resp. decreasing operators, the corollary follows by Theorem 1.5(i). Recall (Corollary 1.4) that both retractions $\bar{h}_{\underline{f}}$ and $\underline{h}_{\bar{f}}$ have X_f as their image set. In case X is a finite lattice each map f (being a member of the finite set X^X) has an idempotent power f^{n_f}. Crapo uses the retraction $r: X^X \to X^X$ defined by $r(f) = f^{n_f}$ to prove that $\mathscr{H}(X)$ is a lattice when X is a finite lattice. Now, since $\underline{f}$ is increasing and $\bar{f}$ is decreasing $\underline{f} \leqslant \underline{f}^n \leqslant f^n \leqslant \bar{f}^n \leqslant \bar{f}$ is satisfied for each n and so, when f^{n_0} is idempotent (i.e., a retraction) for some n_0 it follows by Corollaries 2.3, 3.4 and Theorem 1.5 that $\bar{h}_{\underline{f}} \leqslant f^{n_0} \leqslant \underline{h}_{\bar{f}}$. Thus the retraction $r(f)$ used by Crapo satisfies $\bar{h}_{\underline{f}} \leqslant r(f) \leqslant \underline{h}_{\bar{f}}$. In general, X_f, the image set of $\bar{h}_{\underline{f}}$, $\underline{h}_{\bar{f}}$ is properly contained in $r(f)(X) = X_{f^{n_f}}$.

6

In this section we study closure operators and increasing operators. All results have their dualized analogs for anticlosure and decreasing operators resp.

Each map $f: X \to X$ which satisfies $f \geqslant i$ is increasing and so $\bar{f}$, the retraction associated with f, is a closure operator. The "closed" elements x' of $\bar{f}$ (i.e., those satisfying $x' = \bar{f}(x')$) are the fixed points of f (Theorem 1.4) and hence precisely those $x' \geqslant f(x')$.

Given a family $\{h_i\}$ of closure operators we get by Theorem 1.3 and the discussion above that $\bigwedge h_i$ $(\geqslant i)$ is both decreasing and increasing and hence (being a retraction) a closure operator. Thus

COROLLARY 1.6. *Let X be a complete lattice. Then $\mathscr{C}(X)$, the set of closure operators on X, is a complete $\bigwedge$-sublattice of X^X in the pointwise partial order. Moreover, $\mathscr{C}(X)$ is a complete sublattice of $\mathscr{H}(X)$.*

Our results can now be used to characterize the *transitive closure* $\hat{f}$, [3, p. 23] of the map $f: X \to X$, i.e., the least closure operator (see Corollary 1.6) greater than f. Since $\hat{f} \geqslant f \vee i \geqslant i$ must be satisfied and since $f \vee i$ is increasing it follows that $\hat{f} = \overline{f \vee i} = \bar{h}_{f\vee i}$, that is to say, $\hat{f}$ is the retraction associated with $f \vee i$ (Corollary 2.3). The elements x' closed under $\hat{f}$ are the fixed points of $f \vee i$ and thus are precisely those $x' \geqslant f(x')$.

When h_1, h_2, are retractions on X then neither their composites h_1h_2, h_2h_1, nor their join $h_1 \vee h_2$ need be retractions. Note however that when $h_1 \leqslant h_2$ it follows by Theorem 1.5(ii) that h_1h_2, h_2h_1 are retractions (compare with [7, Theorem 20]). It is observed in [12] that when h_1, h_2 are closure operators then $h_1 \vee h_2$ is a closure operator iff $h_1h_2 = h_2h_1$. Theorem 23 [7] states that a necessary and sufficient condition for h_1h_2 to be a closure operator is that $h_1h_2 = \widehat{h_1 \vee h_2}$. Both h_1h_2 and h_2h_1 are closure operators iff h_1 and h_2 commute. Also h_1h_2 is a closure operator iff the h_1-closure of each h_2-closed element is h_2-closed, i.e., $h_2h_1h_2 = h_1h_2$ [7]. These and related results (see e.g., [3, Theorem 4, p. 13]) can now be extended and clarified using the notion of the decreasing closure and Theorem 1.4. Most of the results actually hold for arbitrary maps f_1, f_2 provided that they satisfy $f_1 \vee f_2 \leqslant f_1f_2$, $f_1 \vee f_2 \leqslant f_2f_1$, with retraction operators and decreasing closures replacing closure operators and transitive closures.

THEOREM 1.6. *Let f_1, f_2 be maps on the complete lattice X satisfying $f_1 \vee f_2 \leqslant f_1f_2$. Then* (i) *$f_1 \vee f_2$ and f_1f_2 are increasing operators*; (ii) $\overline{f_1 \vee f_2} = \overline{f_1f_2}$; (iii) $X_{f_1\vee f_2} = X_{f_1f_2}$.

Proof. (i) follows by $f_1 \vee f_2 \leqslant f_1f_2 \leqslant (f_1 \vee f_2)^2 \leqslant (f_1f_2)^2$. Hence $f_1 \vee f_2 \leqslant f_1f_2 \leqslant \overline{f_1 \vee f_2}$, where $\overline{f_1 \vee f_2}$ is the retraction associated with $f_1 \vee f_2$. By Theorems 1.4, 3.4, $\overline{f_1f_2} = \overline{f_1 \vee f_2}$ and $X_{f_1\vee f_2} = X_{f_1f_2}$, proving (ii), (iii).

COROLLARY 2.6. *If in Theorem 2.5 $f_1 \vee f_2 \leqslant f_2f_1$ is also satisfied then $\overline{f_1 \vee f_2} = \overline{f_1f_2} = \overline{f_2f_1}$ and $X_{f_1\vee f_2} = X_{f_1f_2} = X_{f_2f_1} = X_{f_1} \cap X_{f_2}$.*

Proof. We have only to prove that $X_{f_1\vee f_2} = X_{f_1} \cap X_{f_2}$. $X_{f_1} \cap X_{f_2} \subseteq X_{f_1\vee f_2}$ is obvious. Now if $(f_1 \vee f_2)(x_0) = x_0$ then $f_1(x_0) \leqslant x_0$, $f_2(x_0) \leqslant x_0$. Hence $x_0 = (f_1 \vee f_2)(x_0) \leqslant f_2f_1(x_0) \leqslant f_2(x_0) \leqslant x_0$ and $f_2(x_0) = x_0$ follows. Similarly, $f_1(x_0) = x_0$ is proved and so $X_{f_1\vee f_2} \subseteq X_{f_1} \cap X_{f_2}$, completing the proof.

Theorems 1.6, 2.6 and Corollary 2.6 obviously hold for retraction operators h_1, h_2 which satisfy $h_1 \vee h_2 \leqslant h_1h_2$, $h_1 \vee h_2 \leqslant h_2h_1$. In particular when h_1, h_2 are any given closure operators then by $i \leqslant h_1$, $i \leqslant h_2$, $h_1 \vee h_2 \leqslant h_1h_2$, $h_1 \vee h_2 \leqslant h_2h_1$ is always satisfied. Noting that when $f \geqslant i$ then $\bar{f} = \hat{f}$, the transitive closure of f, we get

COROLLARY 3.6. *Let h_1, h_2 be closure operators on the complete lattice X. Then $\widehat{h_1 \vee h_2} = \widehat{h_1h_2} = \widehat{h_2h_1}$ and $X_{\widehat{h_1 \vee h_2}} = X_{h_1 \vee h_2} = X_{h_1h_2} = X_{h_2h_1} = X_{h_1} \cap X_{h_2}$.*

The proof of the following theorem is based on Theorems 3.4, 1.6 and Corollary 2.6 and is omitted.

THEOREM 2.6. *Let f_1, f_2 be given maps on the complete lattice X which satisfy $f_1 \vee f_2 \leqslant f_1f_2$, $f_1 \vee f_2 \leqslant f_2f_1$.*

(i) *f_1f_2 is a retraction operator iff $f_1f_2 = \overline{f_1 \vee f_2}$;*

(ii) *if $f_1 \vee f_2$ is a retraction operator then $f_1f_2 = f_2f_1 = f_1 \vee f_2$;*

(iii) *if both f_1f_2 and f_2f_1 are retraction operators then $f_1f_2 = f_2f_1$;*

(iv) *if f_1f_2 is a retraction then $f_2f_1 \leqslant f_1f_2 = \overline{f_1 \vee f_2} = (f_1 \vee f_2)^2 = (f_2f_1)^2 = f_2f_1f_2 = f_1f_2f_1$.*

Notice that when h_1, h_2 are retraction operators satisfying $h_1 \vee h_2 \leqslant h_1h_2$, $h_1 \vee h_2 \leqslant h_2h_1$, then h_1h_2 and h_2h_1 are retractions iff $h_1h_2 = h_2h_1$. This gives a partial answer to [6, Problem 6].

7

In this section we study the notion of a u–v connection between the posets X, Y.

DEFINITION 1.7. Let $u\colon X \to X$ be an increasing operator, $v\colon Y \to Y$ be a decreasing operator, where X, Y are posets. The pair $\langle f, g\rangle$ of maps $f\colon X \to Y$ $g\colon Y \to X$ is called a *u–v connection* between X and Y when $gf \geqslant u$ and $fg \leqslant v$.

Obviously, $i_X - i_Y$ connections are the ordinary (dual) Galois connections. From now on let u, v denote fixed maps $u\colon X \to X$, $v\colon Y \to Y$, such that u is an increasing operator and v is a decreasing operator.

The map $f\colon X \to Y$ is *admissible* (*coadmissible*) with respect to the pair (u, v) when $fu \leqslant vf$ ($vf \leqslant fu$, resp.). The map $g\colon Y \to X$ is *admissible* (*coadmissible*) with respect to (u, v) when $gv \leqslant ug$ (resp., $ug \leqslant gv$).

Notice that each map $f\colon X \to Y$, $g\colon Y \to X$ is both admissible and coadmissible with respect to (i_X, i_Y).

LEMMA 1.7. *Let $\langle f, g\rangle$ be a u–v connection between X and Y. Then f is admissible and g is coadmissible with respect to (u, v).*

Proof. Since $gf \geqslant u$, $fg \leqslant v$ and f, g are isotone maps it follows that $fu \leqslant fgf \leqslant vf$, and $ug \leqslant gfg \leqslant gv$.

LEMMA 2.7. *Let $f: X \to Y$ $(g: Y \to X)$ be admissible (coadmissible, resp.) with respect to (u, v). Then the map $\tilde{f} = vfu: X \to Y$ $(\tilde{g} = ugv: Y \to X$, resp.) is both admissible and coadmissible with respect to (u, v) and $\tilde{\tilde{f}} = \tilde{f}$ $(\tilde{\tilde{g}} = \tilde{g}$, resp.).*

Proof. Since $fu \leqslant vf$, $u \leqslant u^2$, $v^2 \leqslant v$ it follows that

$$\tilde{f}u = vfu \cdot u \leqslant v \cdot vfu = v\tilde{f} \leqslant vfu = \tilde{f} \leqslant vfu^2 = \tilde{f}u.$$

Thus $\tilde{f}$ is both admissible and coadmissible with respect to (u, v) and $\tilde{f} = \tilde{f}u = v\tilde{f} = v\tilde{f}u = \tilde{\tilde{f}}$ holds. The other part of the theorem is proved analogously.

As we now show, when $\langle f, g\rangle$ is a u–v connection, then although the maps gf, fg themselves need not in general be retraction operators, there are retraction operators "naturally" associated with them, namely, $\tilde{g}\tilde{f}$ and $\tilde{f}\tilde{g}$ (obviously identical with gf, fg when $(u, v) = (i_X, i_Y)$).

THEOREM 1.7. *Let $\langle f, g\rangle$ be a u–v connection between X and Y. Then*

(i) $\tilde{f} = vfu = vfgfu = vfugfu = vfgvfu = vfugvfu$, $\tilde{g} = ugv = ugfgv = ugfugv = ugvfgv = ugvfugv$;

(ii) $\tilde{f}\tilde{g}\tilde{f} = \tilde{f}$, $\tilde{g}\tilde{f}\tilde{g} = \tilde{g}$;

(iii) *$\langle \tilde{f}, \tilde{g}\rangle$ is a u-v connection between X and Y such that both $\tilde{g}\tilde{f} = ugvfu$ and $\tilde{f}\tilde{g} = vfugv$ are retraction operators;*

(iv) *$ugfu$ is an increasing operator and $u \leqslant ugfu \leqslant \tilde{g}\tilde{f}$, $vfgv$ is a decreasing operator and $v \geqslant vfgv \geqslant \tilde{f}\tilde{g}$.*

Proof. (i) and (ii) are proved by using Lemma 1.4 together with the fact that u is increasing and v is decreasing. For example $\tilde{f} = vfu \leqslant vfu^2 \leqslant vfgfu \leqslant v^2fu \leqslant vfu = \tilde{f}$, proving that $\tilde{f} = vfgfu$; also $\tilde{f} = vfu \leqslant vfu^2gfu^2 \leqslant vfu^2gv^2fu = \tilde{f}\tilde{g}\tilde{f} \leqslant v^2fgv \cdot v^2fu \leqslant vfu = \tilde{f}$, proving that $\tilde{f}\tilde{g}\tilde{f} = \tilde{f}$. All other equalities follow similarly. Now by (ii) we get $\tilde{g}\tilde{f}\tilde{g}\tilde{f} = \tilde{g}\tilde{f}$ and $\tilde{f}\tilde{g}\tilde{f}\tilde{g} = \tilde{f}\tilde{g}$. Thus both $\tilde{g}\tilde{f}$ and $\tilde{f}\tilde{g}$ are retraction operators. To complete the proof of (iii) note that $u \leqslant u^2gfu^2 \leqslant ugv^2fu = \tilde{g}\tilde{f} \leqslant ugvfu \leqslant u^2gvfu \leqslant ugv^2fu = \tilde{g}\tilde{f}$. The proof that $v \geqslant \tilde{f}\tilde{g} = vfugv$ is analogous. To show that $ugfu$ is an increasing operator note that $ugfu] \cdot [ugfu \geqslant ugfu^4 [\geqslant ugfu$. All other statements of (iv) follow similarly, completing the proof.

Note that in Theorem 1.7 (iv) we are unable to prove that $ugfu$ is a retraction operator. By Theorem 1.7 it thus follows that every u–v connection generates a connection in the sense of [6], namely, $\langle \tilde{f}, \tilde{g} \rangle$. Conversely, each connection $\langle f, g \rangle$ in the sense of [6] is an fg–gf connection. Hence, using [6, Theorem 2] we get:

THEOREM 2.7. *Let $\langle f, g \rangle$ be a u–v connection between the posets X and Y. Then the sets $vfu(X) = \tilde{f}(X)$ and $ugv(Y) = \tilde{g}(Y)$ are isomorphic posets in the partial order induced by Y, X resp. $\tilde{f}(X)$, $\tilde{g}(Y)$ are retracts of Y, X, resp. Consequently, for X, Y complete lattices, $\tilde{f}(X)$ and $\tilde{g}(Y)$ are isomorphic complete lattices.*

Concerning the question of uniqueness we have:

THEOREM 3.7. *Let $\langle f, g \rangle$ and $\langle f^*, g \rangle$ be u–v connections between X and Y. Then $\tilde{f} = \tilde{f}^*$. Consequently, both connections generate the same retraction operators* (*Theorem* 1.7 (iii)).

Proof. By $gf \geqslant u$, $fg \leqslant v$, $gf^* \geqslant u$, $f^*g \leqslant v$ we get $\tilde{f} = vfu \leqslant vfu^2 \leqslant vfgf^*u \leqslant v^2f^*u \leqslant vf^*u = \tilde{f}^*$. $\tilde{f}^* \leqslant \tilde{f}$ follows similarly, completing the proof.

The set of all maps $f_i : X \to Y$ for which $\langle f_i, g \rangle$ is a u–v connection for some fixed map g: $Y \to X$ is denoted by $g^{\#}_{(u,v)}$. By the definition of a u–v connection together with Lemma 2.7 and Theorem 3.7 we obtain:

COROLLARY 1.7. *Let g: $Y \to X$ be given such that $g^{\#}_{(u,v)} \neq \varnothing$. Then*

(i) *if $f_i, f_j \in g^{\#}_{(u,v)}$ then $f_i g f_j \in g^{\#}_{(u,v)}$;*

(ii) *if $f_i, f_j \in g^{\#}_{(u,v)}$ then $\tilde{f}_i = \tilde{f}_j$, and $\tilde{f}_i$ is the only map in $g^{\#}_{(u,v)}$ which is both admissible and coadmissible with respect to (u, v).*

When dealing with the composition of u–v connections some conditions (automatically fulfilled when dealing with Galois ($i_X - i_Y$) connections) have to be imposed on the maps involved. In the next theorem X, Y, and Z denote posets and u: $X \to X$, $u_1 : Y \to Y$ are increasing operators, while v: $Y \to Y$, $v_1 : Z \to Z$ are decreasing operators.

THEOREM 4.7. *Let $\langle f, g \rangle$ be a u–v connection between X and Y and let $\langle p, q \rangle$ be a u_1–v_1 connection between Y and Z. Then*

(i) *$\langle pf, gq \rangle$ is a u–v_1 connection between X and Z provided the following conditions are satisfied: $fu \leqslant u_1 f$, $ug \leqslant gu_1$, $pv \leqslant v_1 p$, $vq \leqslant qv_1$.*

(ii) *The retractions $\widetilde{gq}\,\widetilde{pf}$, $\widetilde{pf}\,\widetilde{gq}$ (Theorem 1.7 (iii)) are given by $\widetilde{gq}\,\widetilde{pf} = ugu_1qv_1pu_1fu$, $\widetilde{pf}\,\widetilde{gq} = v_1pvfugvqv_1$.*

Proof. Since $gq\,pf \geqslant gu_1f \geqslant gfu \geqslant u^2 \geqslant u$, and $pf\,gq \leqslant pvq \leqslant v_1pq \leqslant v_1{}^2 \leqslant v_1$, (i) follows. (ii) follows by an appropriate use of the conditions given above together with Lemma 1.7, and the fact that u, u_1 are increasing while v, v_1 are decreasing operators.

Since each connection [6] is also a connection in the sense of Definition 1.7, Theorem 4.7 gives a partial answer to [6, Problem 3].

Let $X \circledast_{(u,v)} Y \subseteq Y^X$ denote the set of all maps $f: X \to Y$ which are part of a u–v connection $\langle f, g\rangle$ between the posets X and Y. Clearly, $X \circledast_{(i_X, i_Y)} Y = X \circledast Y$, the set of residuated maps $f: X \to Y$. When X, Y are bounded posets $X \circledast_{(u,v)} Y$ is not empty. For example, the map $O: X \to Y$ satisfying $O(x) = 0$ for each $x \in X$ belongs to $X \circledast_{(u,v)} Y$ since $\langle O, I\rangle$ is a u–v connection where $I: Y \to X$ is defined by $I(y) = 1$ for each $y \in Y$. In order to characterize the elements $f \in X \circledast_{(u,v)} Y$ notice that by Lemma 1.7 f must be admissible with respect to (u, v). Also, for $\tilde{f} = vfu$, $\tilde{g} = ugv$, $\tilde{g}\tilde{f}u \geqslant u$ and $\tilde{f}\tilde{g}v \leqslant v$ (Theorem 1.7 (iii)) and thus the pair $\langle \tilde{f}, \tilde{g}\rangle$ of (restricted) maps $\tilde{f}: u(X) \to v(Y)$, $\tilde{g}: v(Y) \to u(X)$ is a Galois connection between the posets $u(X)$ and $v(Y)$. This proves the "only if" part of

THEOREM 5.7. *The map $f: X \to Y$ belongs to $X \circledast_{(u,v)} Y$ iff*

(i) $fu \leqslant vf$;

(ii) *the map $\tilde{f} = vfu: u(X) \to v(Y)$ is residuated, i.e., $\tilde{f} \in u(X) \circledast v(Y)$.*

To prove the "if" part assume that (i), (ii) are fulfilled. By (ii) a (unique!) map $g^*: v(Y) \to u(X)$ exists such that $\langle \tilde{f}, g^*\rangle$ is a Galois connection between $u(X)$ and $v(Y)$, i.e., $g^*\tilde{f}u \geqslant u$, $\tilde{f}g^*v \leqslant v$. By naturally extending g^* to a map $g = ug^*v: Y \to X$ it follows using (i) that $gf = ug^*vf \geqslant ug^*v^3f \geqslant ug^*vfu^2 = ug^*\tilde{f}u \geqslant u \cdot u \geqslant u$ and $fg = fug^*v \leqslant fu^2g^*v \leqslant vfug^*v = \tilde{f}g^*v \leqslant v$. Thus $\langle f, g\rangle$ is a u–v connection and $f \in X \circledast_{(u,v)} Y$.

In a way, Theorem 5.7 characterizes u–v connections modulo Galois connections between the "reduced" posets $u(X)$ and $v(Y)$. We conclude our investigation by stating:

THEOREM 6.7. *Let X, Y be complete lattices. Then $X \circledast_{(u,v)} Y$, ordered by the pointwise partial order is a complete $\bigvee$-sublattice of Y^X.*

Proof. The map $O: X \to Y$ defined by $O(x) = 0$ for each $x \in X$ belongs to $X \circledast_{(u,v)} Y$ and is its least element. When $\{f_i\} \subseteq X \circledast_{(u,v)} Y$ is given, such that $\langle f_i, g_i\rangle$ are u–v connections between X and Y then $(\bigwedge g_i)(\bigvee f_i) = \bigwedge g_i(\bigvee f_i) \geqslant \bigwedge g_if_i \geqslant u$, $(\bigvee f_i)(\bigwedge g_i) = \bigvee f_i(\bigwedge g_i) \leqslant \bigvee f_ig_i \leqslant v$. Thus $\langle \bigvee f_i, \bigwedge g_i\rangle$ is a u–v connection and hence $\bigvee f_i \in X \circledast_{(u,v)} Y$. This proves (see [4, p. 112]) that $X \circledast_{(u,v)} Y$ is a complete lattice, which is of course a complete $\bigvee$-sublattice of Y^X.

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